Tutorial 2
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BACK

Questions of today

1. If $f \in \mathcal{F}$, show that
   - The Fourier transform of $f(x+h)$ is $\hat{f}(\xi)e^{2\pi ih\xi}$.
   - The Fourier transform of $f(x)e^{-2\pi ixh}$ is $\hat{f}(\xi+h)$.
   - The Fourier transform of $f(cx)$ is $\hat{c}^{-1}f(c^{-1}\xi)e^{2\pi h\xi}$ for $c > 0$.
   - The Fourier transform of $f'(x)$ is $2\pi i\xi\hat{f}(\xi)$.
   - The Fourier transform of $-2\pi ixf(x)$ is $(\hat{f})'(\xi)$.
2. If $0 \neq f \in \mathcal{F}$ has the property that $\hat{f} = cf$ for some constant c. Show that $c^4 = 1$.
3. Let w_1, w_2 be two complex numbers that are linearly independent over $\mathbb{R}$. Show that if f is an entire function that satisfies $f(z+w_1) = f(z+w_2) = f(z)$ for any $z \in \mathbb{C}$. Then f is a constant function.
4. Find the Fourier transform formula to $f(x) = \frac{t}{\pi(x^2+t^2)}$
5. Suppose $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{F}$. Show that
$$a_n = \frac{(2\pi i)^n}{n!}\int_{-\infty}^{\infty} \hat{f}^{\xi}\xi^n d\xi$$
6. Let
$$\Theta(z|\tau) = \sum_{n=-\infty}^{\infty} e^{\pi in^2\tau}e^{2\pi inz}.$$
   Show that
   - $\Theta(z+1|\tau) = \Theta(z|\tau)$
   - $\Theta(z+\tau|\tau) = e^{-\pi i\tau}e^{-2\pi iz}\Theta(z|\tau)$

Hints & solutions of today

1. By direct calculation, make sure you understand why the functions are of class $\mathcal{F}$ for each case.
2. Compare the Fourier transform and inverse Fourier transform, one finds that taking Fourier transform twice is then same as applying the transform $x \leftrightarrow -x$.
3. Each $z \in \mathbb{C}$ can be written as $aw_1 + bw_2$ for some unique real numbers a, b. Choose integers m, n so that $|a-m|, |b-n| < 1$,then we have
$$f(z) = f((a-m))w_1 + (b-n)w_2).$$
   This shows that $f(\mathbb{C}) = f(S)$, where S is the the compact set
$$\{pw_1 + qw_2 \in \mathbb{C} : -1 \leq p, q \leq 1\}.$$
   Therefore $f(\mathbb{C})$ is compact, and hence bounded.

   Remark We can get the same conclusion if we only assume w_1, w_2 are linearly independent over $\mathbb{Q}$. The only case remains is when w_1 is an irrational multiple of w_2, but then we can show that the set
$$\{z \in \mathbb{C} : f(z) = f(0)\}$$
   has an accumulation at the origin.
4. This is homework 1.
5. Note that $a_n = \frac{f^{(n)}(0)}{n!}$. Then find the $f^{(n)}$ using Question 1.
6. Direct calculation.

Outline

In this note, we have

1. An alternative way to find ther Fourier transform of $e^{-\pi x^2}$.
2. A discussion of Theta functions.

Fourier transform of $e^{-\pi x^2}$

The function $e^{-\pi x^2}$ will appear many times in the course, we first recall how to integrate it on the whole real line, using techniques that we learned in Advanced Calculus II.

$$\begin{aligned}(\int_{-\infty}^{\infty} e^{-\pi x^2}dx)^2 &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\pi(x^2+y^2)}dxdy \\ &= \int_o^{2\pi}\int_0^{\infty} e^{-\pi r^2}rdrd\theta \\ &= -2\pi \cdot \frac{1}{2\pi}e^{-\pi r^2}|_0^{\infty} \\ &=1.\end{aligned}$$

We see that $\int_{-\infty}^{\infty} e^{-\pi x^2}dx = 1$, since it must be positive.

Now, we will use Question 1 to calculate the Fourier transform of $e^{-\pi x^2}dx$. To begin with, we let $F(\xi)$ be its Fourier transform:

$$F(\xi) = \int_{\infty}^{\infty} e^{-\pi x^2}e^{-2\pi ix\xi}dx.$$

We differentiate both sides with respect to ξ and obtain:

$$\begin{aligned}F'(\xi) &= i\int_{-\infty}^{\infty}(-2\pi xe^{-\pi x^2})e^{-2\pi ix\xi}dx \\ &= i\int_{-\infty}^{\infty}(\frac{d}{dx}e^{-\pi x^2})e^{-2\pi ix\xi}dx \\ &= -2\pi\xi F(\xi).\end{aligned}$$

Where in the last equality, we use Question 1. If you know some ODE, then we will see that $F(\xi) = Ae^{-\pi\xi^2}$. Otherwise, we can let G to be the function:

$$G(\xi) = F(\xi)e^{\pi\xi^2}.$$

Differentiation gives:

$$\begin{aligned}G'(\xi) &= F'(\xi) \cdot e^{\pi\xi^2} + F(\xi) \cdot \frac{d}{d\xi}e^{\pi\xi^2} \\ &= -2\pi\xi F(\xi) \cdot e^{\pi\xi^2} + F(\xi) \cdot 2\pi\xi e^{\pi\xi^2} \\ &= 0.\end{aligned}$$

Therefore, G is a constant A, and $F(\xi) = Ae^{-\pi\xi^2}$.
Substituting, $\xi = 0$, we find

$$A = F(0) \triangleq \int_{-\infty}^{\infty} e^{-\pi\xi^2} = 1.$$

Therefore, $F(\xi) = e^{-\pi\xi^2}$.

Some remarks about Question 4 and 6.

Let w_1, w_2 be two complex numbers which are linearly independent over $\mathbb{R}$.

Definition: A function $f : \mathbb{C} \to \mathbb{C}$ is said to be doubly-periodic with respect to periods w_1, w_2 if
$$f(z+w_1) = f(z+w_2) = f(z)$$
for any $z \in \mathbb{C}$.

We recall that $\mathbb{C}$ with the usual addition is an abelian group. The subgroup Λ of $\mathbb{C}$ generated by w_1 and w_2 is isomorphic to $\mathbb{Z}^2$. Note that doubly periodic functions are exactly those functions that descends to a map $\mathbb{C}/\Lambda \to \mathbb{C}$.

From Question 4, we see that there is no non trivial doubly periodic holomorphic function. However, there do exist many doubly periodic meromorphic functions, as one may find some in chapter 9 of the textbook.
The functions of Question 6 is a holomoprhic functions that are close to being doubly periodic. In fact, part one of the question says 1 is a period of $\Theta(z|\tau)$, and part two says that $\Theta(z|\tau)$ has some simply transformation formula in the τ direction. Form now on, we take $w_1 = 1, w_2 = \tau$, so

$$\Lambda = \{p + q\tau : p, q \in \mathbb{Z}\}.$$

Lemma 1 : Let $\lambda = p + q\tau \in \Lambda$, then
$$\Theta(z+\lambda|\tau) = e^{-\pi iq^2\tau - 2\pi iqz}\Theta(z|\tau).$$

Assume without loss of generality that $q \geq 0$. The case that $q = 0$ is part one of Question 6. Now assume it is true for $q \leq k$. Let $q = k+1$, we use part two of Question 6:

$$\begin{aligned}\Theta(z+\lambda|\tau) &= \Theta(z+p+(k+1)\tau|\tau) \\ &= e^{-\pi i\tau - 2\pi i(z+p+k\tau)}\Theta(z+p+k\tau|\tau) \\ &= e^{-\pi i\tau - 2\pi i(z+p+k\tau)}e^{-\pi ik^2\tau - 2\pi ikz}\Theta(z|\tau) \\ &= e^{-\pi i(k+1)^2\tau - 2\pi i(k+1)z}\Theta(z|\tau).\end{aligned}$$

For $\lambda = p + q\tau$, let $e(\lambda, z) = e^{-\pi iq^2\tau - 2\pi iqz}$. We thus have

$$\Theta(z+\lambda|\tau) = e(\lambda, z)\Theta(z|\tau).$$

Lemma 2 :
$$e(\lambda_1 + \lambda_2, z) = e(\lambda_1, z + \lambda_2)(\lambda_2, z).$$

This can be proved similar as above by direct computation. We remark that we can define an action $\cdot_\tau$ of Λ on $\mathbb{C} \times \mathbb{C}$ by

$$\lambda \cdot_\tau (z, t) = (z + \lambda, e(\lambda)t)$$

Lemma 2 says exactly that this is well defined as a group action. On the other hand, we can define a map

$$s : \mathbb{C} \to \mathbb{C} \times \mathbb{C}$$

by

$$s(z) = (z, \Theta(z|\tau)).$$

Lemma 1 says exactly that this map is Λ equivariant.
In this final part, we will see why $\Theta(z|\tau)$ are analogue of doubly periodic functions.
We can define another action $\cdot$ of Λ on $\mathbb{C} \times \mathbb{C}$ by

$$\lambda \cdot (z, t) = (z + \lambda, t)$$

The following three data are equivalent:

1. A doubly periodic functions $f : \mathbb{C} \to \mathbb{C}$.
2. A function $s : \mathbb{C} \to \mathbb{C} \times \mathbb{C}$ of the form $s(z) = (z, f(z))$ that is Λ equivariant (with respect to $\cdot$).
3. A function $\mathbb{C}/\Lambda \to \mathbb{C}$.

The function $\Theta(z|\tau)$ satisfies the anologoe of the condition 2 if we considered $\mathbb{C} \times \mathbb{C}$ as equipped with the Λ action $*_\tau$. Therefore, functions like $\Theta(z|\tau)$ are also called twisted functions of $\mathbb{C}/\Lambda$ (which can be topologized as a torus). We see that there is no holomorphic functions on $\mathbb{C}/\Lambda$ (except the trivial one!), but there can be (nontrivial) holomorphic twisted functions on $\mathbb{C}/\Lambda$.